The Strophoids.

BY WILLIAM WOOLSEY JOHNSON, *Annapolis, Md.*

THE term *Strophoid* has been applied by French writers to a cubic curve, of which the symmetrical form has been discussed by Dr. James Booth under the name of the Logocyclic Curve. As this curve is one of the class considered in this paper, and as the term *Strophoid* is appropriate to the mode of generation of the whole class, I have ventured to use the word in a more extended signification, and define the strophoid as the *locus of the intersection of two straight lines which rotate uniformly about two fixed points in a plane.*

A and B being the fixed points, if we denote by θ and ϕ, the inclinations of the radii-vectores PA and PB, the direction AB being taken as that of the prime vector, we have by the definition

$$n\theta + m\phi = \alpha, \tag{1}$$

in which the ratio $m:n$, which determines the relative velocity of rotation, will in what follows be regarded as commensurable, so that m and n may be taken as integers prime to one another. Restricting m and n to positive values, (1) represents the case in which the lines rotate in opposite directions; and when they rotate in the same direction we may write

$$n\theta - m\phi = \alpha. \tag{2}$$

Taking A as the origin of rectangular co-ordinates, and AB as the axis of x,

$$x = \sqrt{(x^2 + y^2)} \cos\theta, \qquad y = \sqrt{(x^2 + y^2)} \sin\theta;$$

hence, if we put

$$(x + iy)^n = X_n + iY_n, \tag{3}$$

[so that $X_0 = 1$, $Y_0 = 0$; $X_1 = x$, $Y_1 = y$; $X_2 = x^2 - y^2$, $Y_2 = 2xy$, etc.], we have, by De Moivre's Theorem,

$$X_n = (x^2 + y^2)^{\frac{n}{2}} \cos n\theta, \qquad Y_n = (x^2 + y^2)^{\frac{n}{2}} \sin n\theta. \tag{4}$$

From this we have (r and s being positive),

$$X_{r+s} = X_r X_s - Y_r Y_s, \quad Y_{r+s} = Y_r X_s + X_r Y_s, \tag{5}$$

and, if $r > s$,

$$(x^2+y^2)^s X_{r-s} = X_r X_s + Y_r Y_s, \quad (x^2+y^2)^s Y_{r-s} = Y_r X_s - X_r Y_s. \tag{6}$$

If we denote by X'_m and Y'_m the results of putting $x-a$ in place of x in the values of X_m and Y_m, $\cos m\phi$ and $\sin m\phi$ are given by equations similar to (4); and, putting $q = \cot\alpha$, we have from (1)

$$X_n X'_m - Y_n Y'_m - q\,(Y_n X'_m + X_n Y'_m) = 0, \tag{7}$$

when the lines rotate in opposite directions; and from (2)

$$X_n X'_m + Y_n Y'_m - q\,(Y_n X'_m - X_n Y'_m) = 0, \tag{8}$$

when the lines rotate in the same direction.

Differentiating (3) with respect to x, we find

$$\frac{d}{dx}(X_m + iY_m) = m\,(x+iy)^{m-1} = m\,(X_{m-1} + iY_{m-1}),$$

therefore

$$\frac{d}{dx} X_m = mX_{m-1}, \qquad \frac{d}{dx} Y_m = mY_{m-1};$$

hence, developing X'_m and Y'_m,

$$X'_m = X_m - maX_{m-1} + \frac{m\,(m-1)}{1.2}\,a^2 X_{m-2} - \ldots + (-1)^m a^m,$$
$$Y'_m = Y_m - maY_{m-1} + \ldots - (-1)^m m a^{m-1} y.$$

Substituting these values and making use of (5), (7) becomes

$$X_{n+m} - qY_{n+m} - ma[X_{n+m-1} - qY_{n+m-1}] + \ldots + (-1)^m a^m [X_n - qY_n] = 0, \tag{9}$$

the rectangular equation when the lines rotate in opposite directions.

Similarly substituting in (8) and making use of (6), we have, if $n > m$,

$$(x^2+y^2)^m [X_{n-m} - qY_{n-m}] - ma\,(x^2+y^2)^{m-1}[X_{n-m+1} - qY_{n-m+1}] + \ldots$$
$$\ldots + (-1)^m a^m [X_n - qY_n] = 0; \tag{10}$$

but if $n < m$, the first terms of this equation take a different form, and (remembering that $X_0 + qY_0 = 1$), we have

$$(x^2+y^2)^n\left\{X_{m-n}+qY_{m-n}-ma\,[X_{m-n-1}+qY_{m-n-1}]+\ldots\ldots+(-1)^{m-n}\frac{m(m-1)\ldots\ldots(n+1)}{(m-n)!}a^{m-n}\right\}$$
$$+(-1)^{m-n+1}\frac{m(m-1)\ldots\ldots n}{(m-n+1)!}a^{m-n+1}(x^2+y^2)^{n-1}[X_1-qY_1]\ldots\ldots+(-1)^m a^m[X_n-qY_n]=0. \quad (11)$$

(10) and (11) are therefore forms of the rectangular equation when the lines rotate in the same direction.

The expressions $X_n - qY_n$, etc., which occur in these equations, are the products of linear factors which are all real; for, putting

$$X_n - qY_n = 0,$$

we have from (4)

$$\tan n\theta = \tan \alpha,$$

whence

$$\theta = \frac{\alpha}{n},\ \frac{\alpha+\pi}{n},\ \ldots\ldots\ \frac{\alpha+(n-1)\pi}{n},$$

values which determine n different real factors. Thus (9) indicates $n+m$ real asymptotes; and (10) and (11) indicate $n-m$ or $m-n$ real asymptotes, the remaining points at infinity being the circular points. Each of the equations also indicates an n-tuple point at A, at which n real tangents make equal angles with one another.

But the tangents at A and B and the asymptotes are readily determined geometrically as follows: When ϕ is a multiple of π, the line BP coincides with AB and

$$\theta = \frac{\alpha + k\pi}{n}; \qquad (12)$$

unless θ is now a multiple of π (which can only happen when α is a multiple of π) the point P is at A, and the value of θ gives the inclination of a tangent at A. In like manner there are m tangents at B, whose inclinations are

$$\phi = \frac{\alpha + k\pi}{m}. \qquad (13)$$

When $\theta - \phi$ is a multiple of π, the rotating lines are parallel and P is at infinity; hence

$$\theta = \frac{\alpha + k\pi}{n+m} \qquad (14)$$

gives the inclinations of the asymptotes. If we put $AP = r$, $BP = r'$, the distances of the point at infinity from the parallel lines are plainly $rd\theta$ and $r'd\phi$; but from (1)

$$\frac{d\theta}{d\phi} = -\frac{m}{n}, \qquad (15)$$

and since at infinity $r = r'$, these distances have the ratio $m:n$. Therefore every asymptote passes through a point C on AB which divides AB (internally if m and n are positive) so that

$$\frac{AC}{CB} = \frac{m}{n}.$$

If, at a finite point of the curve, we denote the angles between the tangent and the radii-vectores r, r' by ψ, ψ', we have

$$\sin\psi = \frac{rd\theta}{ds}, \qquad \sin\psi' = \frac{r'd\phi}{ds},$$

whence, by (15),

$$\frac{\sin\psi}{\sin\psi'} = -\frac{mr}{nr'}.$$

Therefore, to construct the tangent at a given point P, lay off on PA and PB distances PQ, PR proportional to nPB and mPA respectively; bisect QR in T, then PT is the tangent. These results are, of course, applicable to equation (2) in which m is negative.

When $\alpha = 90°$, the expressions $X_n - qY_n$, etc., in (9), (10), and (11) reduce to X_n, etc., and since X_n is an even function of y, we have a curve symmetrical to AB, which may be called a *right strophoid.*

When $\alpha = 0$, the expressions $X_n - qY_n$, etc., must be replaced by Y_n, etc., and since y is a factor of Y_n, the curve breaks up into the line $y = 0$, and a curve of the $(n + m - 1)^{\text{th}}$ degree, which may be called a *substrophoid.*

The substrophoid is, of course, symmetrical to the axis; the number of tangents at A and at B, as well as of asymptotes, is reduced by unity, and these tangents and asymptotes make equal angles with one another and the axis (which is neither a tangent nor an asymptote). The curve cuts the axis in a point D corresponding to $\theta = 0$, $\phi = 0$. At this point an element of arc subtends angles at A and B proportional to the rates of rotation; that is, D divides AB (internally when m and n have the same sign) so that

$$\frac{AD}{DB} = \frac{n}{m}.$$

When the lines rotate in opposite directions, the curve consists of infinite branches without loops; for it is evident that in passing from one position in which the lines are parallel to the next, one and only one of the lines passes through coincidence with the axis. As special cases, we have from (9), when $n = 1$ and $m = 1$, the rectangular hyperbola

$$x^2 - y^2 - 2qxy - a(x - qy) = 0,$$

A and B being extremities of a diameter. The substrophoid in this case is the straight line

$$2x - a = 0.$$

When $n = 2$, $m = 1$, we have

$$x^3 - 3xy^2 - q(3x^2y - y^3) - a(x^2 - y^2 - 2qxy) = 0;$$

the substrophoid being the hyperbola

$$3x^2 - y^2 - 2ax = 0,$$

of which $AD = \frac{2}{3}a$ is the tranverse axis.

When the lines rotate in the same direction, supposing $n > m$, so that the more rapid rotation takes place at B, it is evident that in passing in either direction from a position in which the lines are parallel, BP will come into coincidence with the axis before AP does, that is, P will arrive at A before it arrives at B. Hence the curve consists of m loops between A and B with infinite branches extending from A to the $m - n$ asymptotes. As special cases, we have from (10), when $n = 1$ and $m = 1$, the circle

$$x^2 + y^2 - a(x - qy) = 0$$

passing through A and B.

When $n = 2$ and $m = 1$, (10) gives

$$(x^2 + y^2)(x - qy) - a(x^2 - y^2 - 2qxy) = 0,$$

which is the cubic alluded to in the first paragraph. The right strophoid in this case is Dr. Booth's Logocyclic Curve

$$(x^2 + y^2)x - a(x^2 - y^2) = 0,$$

and the substrophoid is the circle

$$x^2 + y^2 - 2ax = 0,$$

with centre at B. If $n = 1$ and $m = 2$, we obtain from (11) another equation of the same curve; viz.,

$$(x^2 + y^2)(x + qy - 2a) + a^2(x - qy) = 0:$$

the right strophoid being

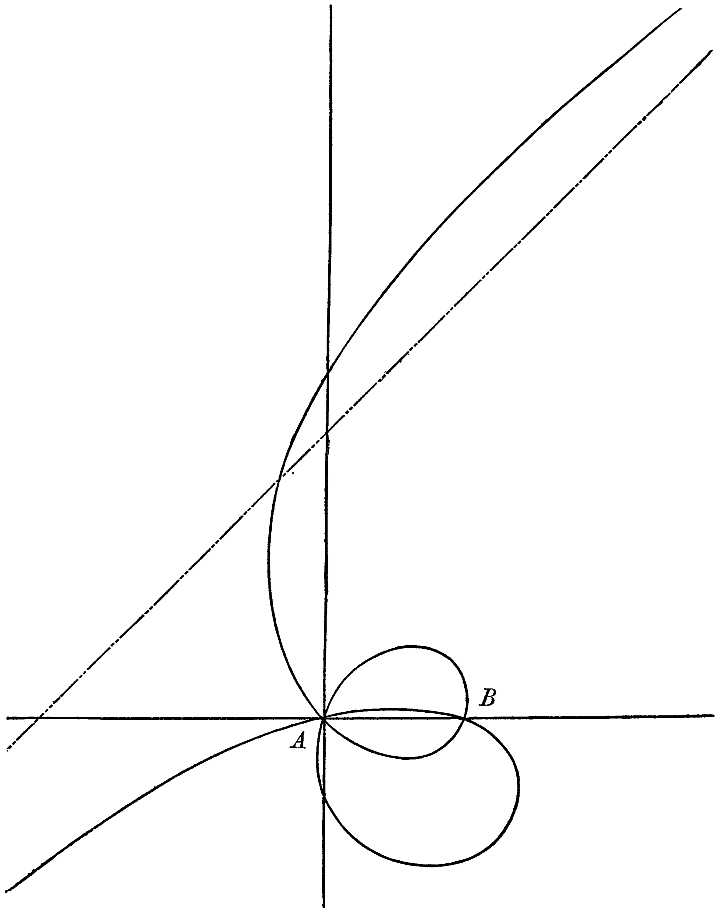
$$(x^2 + y^2)(x - 2a) + a^2x = 0,$$

and the substrophoid the circle

$$x^2 + y^2 - a^2 = 0.$$

The accompanying diagram is constructed for the case in which $n = 3$, $m = 2$ and $\alpha = 45°$, or $3\,\theta - 2\,\phi = \frac{1}{4}\,\pi$; its rectangular equation, therefore, is, from (10),

$$(x^2 + y^2)^2\,(x - y) - 2\,a\,(x^2 + y^2)\,(x^2 - y^2 - 2\,xy) + a^2\,(x^3 - 3\,xy^2 - 3\,x^2y + y^3) = 0.$$



The corresponding substrophoid is a case of the limaçon which is sometimes called the "trisectrix." The mode of employing this curve to trisect an angle is indicated by the equation $3\,\theta = 2\,\phi$.